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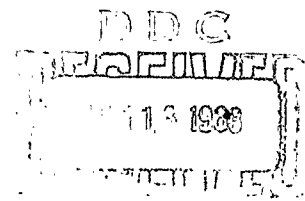
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A Combinatorial Analogue of  
Poincaré's Duality Theorem



Victor Klee

Mathematics Research

May 1963

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### APPENDIX

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A COMBINATORIAL ANALOGUE OF POINCARÉ'S DUALITY THEOREM

by

Professor Victor Klee  
University of Washington

Mathematical Note No. 293

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

May 1963

## 0. Introduction

For a nonnegative integer  $s$  and a finite simplicial complex  $K$ , let  $\beta_s(K)$  denote the  $s$ -dimensional Betti number of  $K$  and let  $f_s(K)$  denote the number of  $s$ -simplices of  $K$ . Our theorem, like Poincaré's, applies to combinatorial manifolds  $M$ , but it concerns the numbers  $f_s(M)$  instead of the numbers  $\beta_s(M)$ . One of the formulae given below is used by the author in [5] to establish a sharp upper bound for the number of vertices of  $n$ -dimensional convex polytopes which have a given number  $i$  of  $(n-1)$  faces. This amounts to estimating the size of the computation problem which may be involved in solving a system of  $i$  linear inequalities in  $n$  variables, and was the original motivation for our study.

A combinatorial  $n$ -manifold is a finite simplicial  $n$ -complex  $M^n$  such that for each  $s$ -simplex  $\sigma^s \in M^n$ , the linked complex  $L(\sigma^s, M^n)$  has the same homology groups as an  $(n-s-1)$ -sphere; analogously, an Eulerian  $n$ -manifold is defined here by the condition that  $L(\sigma^s, M^n)$  always has the same Euler characteristic  $1 - (-1)^{n-s}$  as an  $(n-s-1)$ -sphere, where of course the Euler characteristic of a finite complex  $K$  is the alternating sum  $\chi(K) = \sum_{s=0}^{\infty} (-1)^s f_s(K) (= \sum_{s=0}^{\infty} (-1)^s \beta_s(K))$ . Let  $\mathbb{E}^n$  (resp.  $\mathbb{C}^n$ ) denote the class of all Eulerian (resp. orientable combinatorial)  $n$ -manifold, and for each  $M \in \mathbb{E}^n$  let

$$\beta(M) = (\beta_0(M), \beta_1(M), \dots, \beta_n(M)) \text{ and } f(M) = (f_0(M), f_1(M), \dots, f_n(M)).$$

Then define

$$\beta(\mathbb{C}^n) = \{\beta(M) : M \in \mathbb{C}^n\} \subset \mathbb{R}^{n+1} \text{ and } f(\mathbb{E}^n) = \{f(M) : M \in \mathbb{E}^n\} \subset \mathbb{R}^{n+1}.$$

Poincaré's theorem ( $\beta_s(M) = \beta_{n-s}(M)$ ) implies that the linear span of the set  $\beta(\mathbb{C}^n)$  is an  $\langle(n+2)/2\rangle$ -dimensional subspace of  $\mathbb{R}_+^{n+1}$  (where  $\langle k \rangle$  denotes the greatest integer  $\leq k$ ), and the theorem exhibits a convenient basis for that subspace. The same results are obtained here for the linear span of  $f(\mathbb{E}^n)$ , which has a convenient basis involving binomial coefficients in a simple way. For example, bases for the linear spans of  $f(\mathbb{E}^6) \subset \mathbb{R}_+^7$  and  $f(\mathbb{E}^7) \subset \mathbb{R}_+^8$  are as follows:

$$\mathbb{E}^6: (2,0,0,0,0,0,0), (1,3,2,0,0,0,0), (0,1,4,5,2,0,0), (0,0,1,5,9,7,2);$$

$$\mathbb{E}^7: (1,1,0,0,0,0,0,0), (0,1,2,1,0,0,0,0), (0,0,1,3,3,1,0,0), (0,0,0,1,4,6,4,1).$$

(Note that  $(1,3,2) = (1,2,1) + (0,1,1)$ ,  $(1,4,5,2) = (1,3,3,1) + (0,1,2,1)$ , etc.)

Having a convenient basis for the linear span of  $f(\mathbb{E}^n)$  leads to a useful characterization of the linear relations which must subsist among the numbers  $f_s(M)$  for all  $M \in \mathbb{E}^n$ . It turns out that when  $n = 2u - 1$  (whence  $\chi(M) = 0$  for all  $M \in \mathbb{E}^n$ ) the numbers  $f_n(M)$ ,  $f_{n-1}(M), \dots, f_u(M)$  can be expressed linearly in terms of  $f_{u-1}(M), \dots, f_1(M), f_0(M)$  (the expressions being valid for all  $M \in \mathbb{E}^n$ ), while when  $n = 2u - 2$  the numbers  $f_n(M), f_{n-1}(M), \dots, f_{u-1}(M)$  admit linear expressions in terms of  $f_{u-2}(M), \dots, f_0(M), \chi(M)$ .

Our approach is of a purely combinatorial nature, involving neither subdivision nor homology. The arithmetical identities of §1 are used in §2 to prove the main result, a theorem concerning abstract incidence systems which exhibit some properties of those which are dual to Eulerian manifolds. Applications to Eulerian manifolds and convex polytopes appear in §3.

For the elementary properties of complexes and convex polytopes which are employed here, the reader may consult Alexandroff and Hopf [1] and Weyl [6]. For a treatment of the Euler characteristic which is well suited to the present elementary combinatorial approach, see Hadwiger [3] or Klee [4].

Helpful comments were supplied by C. B. Allendoerfer, E. H. Spanier, and H. S. Zuckerman.

## 1. Some arithmetical identities

This section contains some arithmetical identities involving binomial coefficients, to be employed in §2. Though these may appear in the literature, we have not found them there and thus include their proofs as an aid to the reader. We agree that  $\binom{n}{r}$  is defined in the usual way for all integers  $n$  and  $r$  -- positive, zero or negative (cf. Feller [2, p.40]), and will use freely the basic recursion

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

1.1 PROPOSITION For all nonnegative integers  $j$  and  $k$ ,

$$\sum_{i=0}^j (-1)^i \binom{j}{i} \binom{i+j}{k} = (-1)^j \binom{j}{k-j}.$$

Proof. Let  $V(i, j, k) := (-1)^i \binom{j}{i} \binom{i+j}{k}$ , whence

$$\begin{aligned} V(i, j, k) &= (-1)^i \left[ \binom{j-1}{i} + \binom{j-1}{i-1} \right] \left[ \binom{i+j-1}{k} + \binom{i+j-1}{k-1} \right] = \\ &= V(i, j-1, k) + V(i, j-1, k-1) - (-1)^{i-1} \binom{j-1}{i-1} \left[ \binom{i+j-2}{k} + \binom{i+j-2}{k-1} + \binom{i+j-2}{k-1} + \binom{i+j-2}{k-2} \right] = \\ &= V(i, j-1, k) + V(i, j-1, k-1) - V(i-1, j-1, k) - 2V(i-1, j-1, k-1) - V(i-1, j-1, k-2). \end{aligned}$$

Now let  $U(j, k) := \sum_{i=0}^j V(i, j, k)$ . Corresponding to the five terms  $V(\cdot, j-1, \cdot)$  in the above expression for  $V(i, j, k)$ , we obtain the five bracketed terms in the equation

$$\begin{aligned} U(j, k) &= [U(j-1, k) + V(j, j-1, k)] + [U(j-1, k-1) + V(j, j-1, k-1)] \\ &- [V(-1, j-1, k) + U(j-1, k)] - 2[V(-1, j-1, k-1) + U(j-1, k-1)] - [V(-1, j-1, k-2) + \\ &+ U(j-1, k-2)] = -U(j-1, k-1) - U(j-1, k-2). \end{aligned}$$

Now clearly when  $m = 0$ ,  $U(m, k) = (-1)^m \binom{m}{k-m}$  for all integers  $k$  (positive, negative, or zero). Suppose the same is known for  $m = j - 1$  and consider the case of  $U(j, k)$ . We have

$$\begin{aligned} U(j, k) &= -U(j-1, k-1) - U(j-1, k-2) \\ &= -(-1)^{j-1} \left[ \binom{j-1}{k-j} + \binom{j-1}{k-j-1} \right] = (-1)^j \binom{j}{k-j}, \end{aligned}$$

so the proof of 1.1 is completed by mathematical induction. ||

1.2 PROPOSITION For  $0 < j < k$ ,

$$\sum_{i=2k-2j}^{2k-j} (-1)^i \binom{j}{2k-j-i} \binom{2k-i}{k-1} \binom{i-1}{k-1} = 0.$$

Proof. Let  $h(k, i) = \frac{2k-i}{k} \binom{i-1}{k-1}$  and note that

$$(1) \quad h(k-1, i) = h(k, i+1) - h(k, i).$$

Indeed, (1) asserts that

$$\frac{2k-i-2}{k-1} \binom{i-1}{k-2} = \frac{2k-i-1}{k} \left[ \binom{i-1}{k-1} + \binom{i-1}{k-2} \right] - \frac{2k-i}{k} \binom{i-1}{k-1},$$

reducing at once to

$$\frac{i-k+1}{k-1} \binom{i-1}{k-2} = \binom{i-1}{k-1},$$

which is easily verified.

Now let  $T(k, j) = \sum_{i=2k-2j}^{2k-j} (-1)^i \binom{j}{2k-j-i} h(k, i)$ . We want to show that  $T(k, j) = 0$  whenever  $0 < j < k$ . Since effective summation in the expression for  $T(k, j)$  is over the range  $\max(2k-2j, k) \leq i \leq 2k-j$ , it is easily verified that  $T(k, k-1) = 0$  for  $k \geq 2$ , and in particular  $T(2, m) = 0$  whenever  $0 < m < 2$ . Now suppose it is known that  $T(k-1, m) = 0$



whenever  $0 < m < k - 1$  and that  $T(k, m) = 0$  whenever  $j < m < k$  (where  $j < k - 1$ ). We can show that  $T(k, j) = 0$  and by proving that

$$(2) \quad T(k, j) + T(k, j+1) + T(k-1, j) + T(k-1, j-1) = 0.$$

To verify (2) we note that

$$T(k, j) = \sum_{2k-2j}^{2k-j} (-1)^i \binom{j}{2k-j-i} h(k, i)$$

and

$$T(k, j+1) + \sum_{2k-2j-2}^{2k-j-1} (-1)^i \binom{j+1}{2k-j-i-1} h(k, i),$$

with summation always on  $i$ , and from (1) it follows that

$$T(k-1, j) = \sum_{2k-2j-3}^{2k-j-3} (-1)^{i-1} \binom{j}{2k-j-i-1} h(k, i) - \sum_{2k-2j-2}^{2k-j-2} (-1)^i \binom{j}{2k-j-i-2} h(k, i),$$

$$T(k-1, j-1) = \sum_{2k-2j+1}^{2k-j} (-1)^{i-1} \binom{j-1}{2k-j-i} h(k, i) - \sum_{2k-2j}^{2k-j-1} (-1)^i \binom{j-1}{2k-j-i-1} h(k, i).$$

Then (2) is proved by showing that for each  $i$ , the net coefficient of  $h(k, i)$  on the left side of (2) is equal to zero. For example, when

$2k-2j+1 \leq i \leq 2k-j-3$ , this coefficient is equal to  $(-1)^i$  times the number

$$\binom{j}{2k-j-i} + \binom{j+1}{2k-j-i-1} - \binom{j}{2k-j-i-1} - \binom{j}{2k-j-i-2} - \binom{j+1}{2k-j-i} - \binom{j-1}{2k-j-i-1},$$

which is equal to zero by the basic recursion used in justifying (1).

The other cases are even simpler. ||

## 2. A theorem on certain incidence systems

By the term incidence system we shall mean a finite set  $X$  with an associated incidence function  $\phi$  and dimension function  $\delta$ ;  $\phi$  is a symmetric real-valued function on  $X \times X$  (that is,  $\phi(x,y) = \phi(y,x)$  for all  $x,y \in X$ ) and  $\delta$  is a function on  $X$  to a set of integers. For each element  $y$  of  $X$  and each integer  $i$ , we define

$$\mu_i(y) := \sum_{x \in X, \delta(x)=i} \phi(y,x).$$

In the case of special interest,  $\phi$  assumes only the values 0 and 1 and is thus the characteristic function of an incidence relation (a symmetric subset of  $X \times X$ ); in this case,  $\mu_i(y)$  is merely the number of  $i$ -dimensional elements of  $X$  which are incident to  $y$ .

The characteristic  $\chi(y)$  of an element  $y \in X$  is defined as the alternating sum

$$\chi(y) := \sum_{i=0}^{\delta(y)} (-1)^i \mu_i(y).$$

For  $d \geq 1$ , the system  $(X, \phi, \delta)$  will be called a d-system provided it satisfies the following conditions:

- (i)  $\max\{\delta(x) : x \in X\} = d - 1$ ;
- (ii)  $\chi(y) = 1$  for all  $y \in X$  with  $\delta(y) \geq 0$ ;
- (iii) whenever  $y \in X$  and  $0 \leq \delta(y) \leq i \leq d - 1$ , then  $\mu_i(y) = \binom{d-\delta(y)}{i-\delta(y)}$ .

Note that these conditions are all satisfied when  $X$  is the simplest triangulation of a  $(d - 1)$ -sphere (that is, the system of all proper faces of a  $d$ -simplex),  $\phi(x,y) = 1$  when  $x$  and  $y$  are incident ( $x \subset y$  or  $x \supset y$ )

and  $\delta = 0$  otherwise, and  $\delta$  is the usual dimension function.

2.1 THEOREM Suppose the incidence system  $(X, \varphi, \delta)$  is a  $d$ -system,  
with  $d = 2u - 1$  or  $d = 2u$ . For  $0 \leq s \leq d - 1$ , let  $f_s$  denote the  
number of  $s$ -dimensional elements of  $X$ , and let  $f_d = \frac{1}{2} \sum_{s=0}^{d-1} (-1)^s f_s$ . For  
 $1 \leq j \leq u$ , let  $\gamma_j^d$  denote the  $2u$ -vector  $(\gamma_{j0}^d, \dots, \gamma_{j(2u-1)}^d)$  where

$$\gamma_{js}^d = \begin{cases} (2u-s) \binom{j}{2u-j-s} & \text{when } d = 2u-1 \\ \binom{j}{2u-j-s} & \text{when } d = 2u. \end{cases}$$

Then the vector  $f = (f_0, \dots, f_{2u-1})$  is linearly dependent on the  $u$   
vectors  $\gamma_1^d, \dots, \gamma_u^d$ . Further,  $f_d = 0$  when  $d = 2u$ .

Proof. For  $i$  and  $j$  between 0 and  $d - 1$ , let

$$g_{ij} = \sum_{x, y \in X; \delta(x)=i, \delta(y)=j} \varphi(x, y).$$

Then of course  $g_{ij} = g_{ji}$ . It follows from condition (iii) that  
 $\mu_{\delta(y)}(y) = 1$  for all  $y \in X$ , and then from condition (ii) that

$$\sum_{i=0}^{\delta(y)-1} (-1)^i \mu_i(y) = 1 - (-1)^{\delta(y)}.$$

Using this equation in conjunction with (iii) we see that for

$$1 \leq m \leq d - 1,$$

$$\begin{aligned} (1 - (-1)^m) f_m &= g_{m0} - g_{m1} + \dots + (-1)^{m-1} g_{m(m-1)} \\ &= \binom{d}{d-m} f_0 - \binom{d-1}{d-m} f_1 + \dots + (-1)^{m-1} \binom{d-m+1}{d-m} f_m. \end{aligned}$$

Hence we obtain the following equations  $E_m$  for  $1 \leq m \leq d-1$ :

$$(\text{odd } m) \quad E_m: 0 = \binom{d}{d-m} f_0 - \binom{d-1}{d-m} f_1 + \cdots + \binom{d-m+1}{d-m} f_{m-1} - 2f_m;$$

$$(\text{even } m) \quad E_m: 0 = \binom{d}{d-m} f_0 - \binom{d-1}{d-m} f_1 + \cdots + \binom{d-m+1}{d-m} f_{m-1}.$$

And we have also

$$E_d: 0 = f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} - 2f_d.$$

These equations are redundant, and we will be concerned only with those having odd indices, that is, with

$$E_1: 0 = \binom{d}{d-1} f_0 - 2f_1,$$

$$E_3: 0 = \binom{d}{d-3} f_0 - \binom{d-1}{d-3} f_1 + \binom{d-2}{d-3} f_2 - 2f_3,$$

. . . . .

terminating with  $E_{2u-1}$  or in other words with

$$(\text{odd } d) \quad E_d: 0 = f_0 - f_1 + f_2 - f_3 + \cdots + f_{d-1} - 2f_d$$

or

$$(\text{even } d) \quad E_{d-1}: 0 = df_0 - (d-1)f_1 + (d-2)f_2 - (d-3)f_3 + \cdots + 2f_{d-2} - 2f_{d-1}.$$

For  $1 \leq r \leq u$  and  $0 \leq s \leq 2u-1$ , let  $\beta_{rs}^d$  be the coefficient of  $f_s$  in the equation  $E_{2r-1}$ , where of course  $\beta_{rs}^d = 0$  for  $s > 2r-1$ . The  $u$  vectors  $\beta_r^d = (\beta_{r0}^d, \dots, \beta_{r(2u-1)}^d) \in \mathbb{R}_+^{2u}$ ,  $1 \leq r \leq u$ , are linearly independent because the  $u \times u$  submatrix

$$(\beta_{rs}^d) \quad (1 \leq r \leq u, 1 \leq \text{odd } s \leq 2u-1)$$

is triangular and has exclusively  $-2$ 's along its main diagonal. Let  $L$  denote the  $u$ -dimensional linear subspace of  $\mathbb{R}_+^{2u}$  which is spanned by  $\{\beta_1^d, \dots, \beta_u^d\}$  and let  $L^0$  denote the orthogonal supplement of  $L$ , consisting of all vectors  $\gamma = (\gamma_0, \dots, \gamma_{2u-1}) \in \mathbb{R}_+^{2u}$  such that  $\sum_{s=0}^{2u-1} \beta_{rs}^d \gamma_s = 0$

for  $1 \leq r \leq u$ . Then of course  $f \in L^0$ , and we will show below that  $\{\gamma_1^d, \dots, \gamma_u^d\} \subset L^0$ . Since  $L^0$  is a  $u$ -dimensional linear space and since the  $u$  vectors  $\gamma_j^d (1 \leq j \leq u)$  are easily seen to be linearly independent, it will follow that  $f$  is a linear combination of the  $\gamma_j^d$ 's. This is the first assertion of 2.1. The second assertion of 2.1 is that if  $d = 2u$ , then  $f_d = 0$  or in other words the  $2u$ -vector  $(1, -1, \dots, 1, -1)$  is orthogonal to the  $2u$ -vector  $f = (f_0, \dots, f_{2u-1})$ . For this it suffices (in view of the first assertion) to show that  $(1, -1, \dots, 1, -1)$  is orthogonal to each of the vectors  $\gamma_j^d (1 \leq j \leq u)$ . But recalling the definition of the vectors  $\gamma_j^d$ , we note that if  $d = 2u$  and  $1 \leq j \leq u$ , then

$$\begin{aligned} \sum_{s=0}^{2u-1} (-1)^s \gamma_{js}^d &= \sum_{s=0}^{2u-1} (-1)^s \binom{j}{2u-j-s} = (-1)^{2u-j} \sum_{i=1-j}^{2u-j} (-1)^i \binom{j}{i} \\ &= (-1)^j \sum_{i=0}^j (-1)^i \binom{j}{i} = 0, \end{aligned}$$

where the final equality follows from 1.1 with  $k = 0$ .

To complete the proof we must show that

$$[d, r, j]: \sum_{s=0}^{2u-1} \beta_{rs}^d \gamma_{js}^d = 0 \quad \text{for } 1 \leq r \leq u, 1 \leq j \leq u.$$

Recalling the definition of  $\beta_{rs}^d$ , we see that  $\beta_{r(2r-1)}^d = -2$ , while

$$\beta_{rs}^d = (-1)^s \binom{d-s}{d-2r+1} \quad \text{for } s \neq 2r-1.$$

When  $d = 2u - 1$  and  $2r - 1$  is not between  $2u - 2j$  and  $2u - j$ ,

the left side of  $[d, r, j]$  is given by

$$\begin{aligned}
 \sum_{s=0}^{2u-1} \beta_{rs}^d (2u-s) \binom{j}{2u-j-s} &= \sum_{s=2u-2j}^{2u-j} \beta_{rs}^d (2u-s) \binom{j}{2u-j-s} \\
 &= \sum_{s=2u-2j}^{2u-j} (-1)^s \binom{j}{2u-j-s} (2u-s) \binom{d-s}{d-2r+1} \\
 &= (-1)^{2u-j} \sum_{i=0}^j (-1)^i \binom{j}{i} (i+j) \binom{i+j-1}{d-2r+1},
 \end{aligned}$$

where the last equality comes from the substitution  $i = 2u - j - s$ .

But  $(i+j) \binom{i+j-1}{d-2r+1} = (d-2r+2) \binom{i+j}{d-2r+2}$ , so we can continue the computation with

$$\begin{aligned}
 &= (-1)^{2u-j} (d-2r+2) \sum_{i=0}^j (-1)^i \binom{j}{i} \binom{i+j}{d-2r+2} \\
 &= (-1)^{2u-j} (d-2r+2) (-1)^j \binom{j}{d-2r+2-j} = 0,
 \end{aligned}$$

where the next-to-last equality comes from 1.1 and the final equality results from the fact that  $\binom{j}{d-2r+2-j} = 0$  when  $2r-1$  is not between  $2u-2j$  and  $2u-j$ .

Now suppose  $d = 2u - 1$  but  $2u - 2j \leq 2r - 1 \leq 2u - j$ .

Correcting the preceding computation to account for the special value of  $\beta_{r(2r-1)}^d$ , we see that the left side of  $[d, r, j]$  is equal to

$$\begin{aligned}
 &(d-2r+2) \binom{j}{d-2r+2-j} + \gamma_{j(2r-1)}^d (-2 - (-1)^{2r-1} \binom{d-(2r-1)}{d-2r+1}) = \\
 &= (d-2r+2) \binom{j}{d-2r+2-j} - (2u - (2r-1)) \binom{j}{2u-j-(2r-1)} = 0.
 \end{aligned}$$

Suppose finally that  $d = 2u$ . When  $2r - 1$  is not between

$2u - 2j$  and  $2u - j$ , the left side of  $[d, r, j]$  is given by

$$\begin{aligned} \sum_{s=d-2j}^{d-j} (-1)^s \binom{j}{d-j-s} \binom{d-s}{d-2r+1} &= (-1)^{d-j} \sum_{i=0}^j (-1)^i \binom{j}{i} \binom{i+j}{d-2r+1} = \\ &= (-1)^{d-j} (-1)^j \binom{j}{d-2r+1-j} = 0, \end{aligned}$$

where we have used 2.1 and the fact that  $d - 2r + 1 - j$  is  $< 0$  or  $> j$ .

When  $d - 2j \leq 2r - 1 \leq d - j$ , correction for the special value of  $\beta_{r(2r-1)}^d$  leads again to the value 0, as in the preceding paragraph.

This completes the proof of 2.1. ||

**2.2 COROLLARY** Suppose  $d$  is a positive integer with  $d = 2u - 1$  or  $d = 2u$ , and  $I$  is a set of integers which includes at least one from each of the  $u$  pairs  $\{0, 1\}, \{2, 3\}, \dots, \{2u - 2, 2u - 1\}$ . If two  $d$ -systems  $(X, \varphi, \delta)$  and  $(X', \varphi', \delta')$  are such that  $f_i = f'_i$  for all  $i \in I$  (where the numbers  $f_i$  and  $f'_i$  are as in 2.1), then  $f_s = f'_s$  for  $0 \leq s \leq 2u - 1$ .

Proof. Upon examination of the basis system  $\gamma_1^d, \dots, \gamma_u^d$ , this is seen to follow at once from 2.1. ||

The following is also an immediate consequence of 2.1.

**2.3 COROLLARY** With hypotheses and notation as in 2.1, let  $\mathcal{E}$  denote the set of all vectors  $\xi = (\xi_0, \dots, \xi_{2u-1}) \in \mathbb{R}_+^{2u}$  such that  $\sum_{s=0}^{2u-1} \xi_s f_s = 0$ . Then  $\mathcal{E}$  includes all vectors  $\xi$  such that  $\sum_{s=0}^{2u-1} \xi_s \gamma_{js}^d = 0$  for  $1 \leq j \leq u$ .

The next theorem is the one whose dual (given in 3.2 below) will be applied in [5].

2.4 THEOREM Suppose  $d$  is a positive integer with  $d = 2u - 1$  or  $d = 2u$ , and  $t$  is an integer with  $0 \leq t \leq u - 1$ . Then there is a vector  $\xi^{dt} = (\xi_u^{dt}, \dots, \xi_{2u-1}^{dt})$  such that

$$f_t = \sum_{i=u}^{2u-1} \xi_i^{dt} f_i$$

whenever the numbers  $f_s$  are obtained from a  $d$ -system as in 2.1. In particular,

$$f_0 = \sum_{i=u}^d (-1)^{i-u} 2^{\binom{i-1}{u-1}} f_i \quad \text{when } d = 2u - 1, \text{ and}$$

$$f_0 = \sum_{i=u}^{d-1} (-1)^{i-u} (2 - \frac{1}{u}) \binom{i-1}{u-1} f_i \quad \text{when } d = 2u.$$

Proof. The first assertion of 2.4 is an easy consequence of 2.3. To justify the specific formulae for  $f_0$  it suffices (in view of 2.3) to show that:

$$\text{for } d = 2u - 1 \text{ and } 1 \leq j \leq u, \quad \sum_{i=u}^{2u-1} (-1)^{i-u} 2^{\binom{i-1}{u-1}} \gamma_{ji}^d = -\gamma_{j0}^d;$$

$$\text{for } d = 2u \text{ and } 1 \leq j \leq u, \quad \sum_{i=u}^{2u-1} (-1)^{i-u} (2 - \frac{1}{u}) \binom{i-1}{u-1} \gamma_{ji}^d = -\gamma_{j0}^d.$$

Recalling the formulae for  $\gamma_{js}^d$  (which depend on the parity of  $d$ ), we see that the statements are easily verified when  $j = u$ , while for  $1 \leq j < u$  they both amount to the assertion that

$$\sum_{i=u}^{2u-1} (-1)^i \binom{j}{2u-j-i} (2u-i) \binom{i-1}{u-1} = 0.$$

But here the effective range of summation is only for  $2u - 2j \leq i \leq 2u - j$  (since otherwise  $\binom{j}{2u-j-i} = 0$ ), and the desired conclusion follows from 1.2. ||



### 3. Application to Eulerian manifolds and convex polytopes

A cell-complex is a finite family  $K$  of convex polytopes (the cells of  $K$ ) such that each face of a member of  $K$  is a member of  $K$ , and the intersection of any two members of  $K$  is a face of both. An  $n$ -dimensional cell-complex  $K^n$  will be called a simple  $n$ -manifold provided for  $0 \leq s \leq i \leq n$ , each  $s$ -cell of  $K^n$  is a face of  $\binom{n+1-s}{i-s}$   $i$ -cells of  $K^n$ .

3.1 PROPOSITION Suppose  $K$  is a simple  $n$ -manifold and  $d = n + 1$ . For  $\sigma, \tau \in K$  let  $\varphi(\sigma, \tau) = 1$  when  $\sigma \subset \tau$  or  $\sigma \supset \tau$ , and  $\varphi(\sigma, \tau) = 0$  otherwise. Let  $\delta$  be the usual dimension function. Then  $(K, \varphi, \delta)$  is a  $d$ -system and hence the results 2.1 - 2.4 apply to the numbers  $f_0, \dots, f_d$ , where  $f_s$  is the number of  $s$ -cells of  $K$  for  $0 \leq s \leq n$ , and  $f_d = \sum_{s=0}^n (-1)^s f_s$ .

Proof. Conditions (i) and (iii) (in the definition of a  $d$ -system) are obviously satisfied, and condition (ii) follows from the fact that when a cell-complex is formed in the natural way from a convex polytope, its Euler characteristic must be equal to 1. ||

Now we recall (from the Introduction) the notion of an Eulerian  $n$ -manifold. This is a finite simplicial  $n$ -complex  $M^n$  such that for each  $s$ -simplex  $\sigma \in M^n$ , the Euler characteristic of the linked complex  $L(\sigma^s, M^n)$  is equal to  $1 - (-1)^{n-s}$ . Here, as usual,  $L(\sigma^s, M^n)$  is the set of all simplexes  $\sigma$  of  $M^n$  such that  $\sigma \cap \sigma^s = \emptyset$  and the join of  $\sigma$  and  $\sigma^s$  is a simplex of  $M^n$ .

3.2 THEOREM Let  $E^n$  denote the class of all Eulerian  $n$ -manifolds. For  $M \in E^n$  and  $0 \leq s \leq n$ , let  $f_s(M)$  denote the number of  $s$ -simplices of  $M$  and let  $\chi(M)$  denote the Euler characteristic  $\sum_{s=0}^n (-1)^s f_s(M)$ .

If  $n = 2u - 1$  and  $M \in E^n$ , then  $\chi(M) = 0$  and the  $2u$ -vector  $(f_0(M), \dots, f_n(M))$  is a linear combination of the  $u$  row-vectors of the  $u \times (2u)$  matrix  $J_n$ :

$$\begin{pmatrix} 1 & 1 & & & & & \\ & 1 & 2 & 1 & & & \\ & & 1 & 3 & 3 & 1 & \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \binom{u}{0}, \binom{u}{1}, \binom{u}{2} & \cdot & \cdot & \cdot & \binom{u}{u-2}, \binom{u}{u-1}, \binom{u}{u} \end{pmatrix}$$

(where zeros have been omitted). Further,

$$f_n = \sum_{j=0}^{u-1} (-1)^{u-1-j} \frac{j+1}{u} \binom{n-j-1}{u-1} f_j.$$

If  $n = 2u - 2$  and  $M \in E^n$ , then the  $2u$ -vector  $(\frac{1}{2} \chi(M), f_0(M), \dots, f_n(M))$  is a linear combination of the  $u$  row-vectors of the  $u \times (2u)$  matrix  $J_n$ :

$$\begin{pmatrix} 1 & 2 & & & & & \\ & 1 & 3 & 2 & & & \\ & & 1 & 4 & 5 & 2 & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \binom{u}{0}, \binom{u}{1} + \binom{u-1}{0}, \binom{u}{2} + \binom{u-1}{1}, \cdot & \cdot & \cdot & \binom{u}{u-2} + \binom{u-1}{u-3}, \binom{u}{u-1} + \binom{u-1}{u-2}, \binom{u}{u} + \binom{u-1}{u-1} \end{pmatrix}$$

(where zeros have been omitted). Further,

$$f_n = (-1)^{u+1} \binom{n}{u-1} \chi + \sum_{j=0}^{u-2} (-1)^{u-j-2} \binom{n-j-1}{u-1} f_j.$$

Proof. For  $\sigma, \tau \in M$  let  $\varphi(\sigma, \tau) = 1$  when  $\sigma \subset \tau$  or  $\sigma \supset \tau$ , and  $\varphi(\sigma, \tau) = 0$  otherwise. For each  $\sigma \in M$  let  $\delta(\sigma) = n - \dim \sigma$  where  $\dim$  is the usual dimension function. With  $d = n + 1$ , we claim that  $(M \sim \{\emptyset\}, \varphi, \delta)$  is a  $d$ -system. Since  $\min\{\dim \sigma : \sigma \in M \sim \{\emptyset\}\} = 0$ , condition (i) is evident. To verify condition (ii) we note that if  $\sigma \in M$ , then relative to the system  $(M \sim \{\emptyset\}, \varphi, \delta)$  the characteristic  $\chi(\sigma)$  of  $\sigma$  (in the sense of §2) is the alternating sum  $\sum_{i=0}^{\delta(\sigma)} (-1)^i \mu_i(\sigma)$ , where  $\mu_i(\sigma)$  is the number of simplices  $\tau \in M$  for which  $\sigma \subset \tau$  and  $\delta(\tau) = i$ . Since  $\delta(\tau) = n - \dim \tau$ , each simplex  $\tau \supset \sigma$  contributes  $(-1)^{n-\dim \tau}$  to the formation of  $\chi(\sigma)$ . The choice  $\tau = \sigma$  contributes nothing to the formation of  $\chi_L(\sigma, M)$ , but each  $\tau \in M$  which properly contains  $\sigma$  corresponds to a simplex of dimension  $\dim \tau - \dim \sigma - 1$  whose join with  $\sigma$  is equal to  $\tau$ , and thus with  $s = \dim \sigma$  each such simplex  $\tau$  contributes  $(-1)^{\dim \tau - s - 1}$  to the formation of  $\chi_L(\sigma, M)$ . Since

$$(-1)^{n-\dim \tau} = (-1)^{n-s-1} (-1)^{\dim \tau - s - 1},$$

we have

$$\chi(\sigma) = (-1)^{n-s-1} \chi_L(\sigma, M) + (-1)^{n-s}.$$

But  $M$  is an Eulerian  $n$ -manifold, so  $\chi_L(\sigma, M) = 1 - (-1)^{n-s}$  and

$$\chi(\sigma) = (-1)^{n-s-1} [1 - (-1)^{n-s}] + (-1)^{n-s} = 1.$$

This establishes condition (ii). Condition (iii) follows at once from the relevant definitions in conjunction with the fact that  $M$  is a simplicial complex. Thus  $(M, \varphi, \delta)$  is a  $d$ -system with  $d = n + 1$ . It is then a routine matter to derive the assertions of 3.2 from 2.1 and 2.4. ||

Of course the results 2.2 and 2.3 can also be dualized so as to apply to Eulerian manifolds, but this is immediate and will be left to the reader. We shall describe explicitly the application to convex polytopes, for this will be required in [5].

An  $n$ -dimensional convex polytope  $P$  will be called simplicial provided all of its  $(n - 1)$ -faces are simplices, and it will be called simple provided each of its vertices is on exactly  $n$  edges (or, equivalently, on exactly  $n$   $(n - 1)$ -faces). From the standard polarity theory (Weyl [6]) it follows that if  $P$  is an  $n$ -dimensional convex polytope in  $\mathbb{R}^n$  and  $0 \in \text{int } P$ , then  $P$  is simplicial if and only if the polar body  $P^0$  is simple, where

$$P^0 = \{x \in \mathbb{R}^n : \sup_{y \in P} \sum_{i=1}^n x_i y_i \leq 1\}.$$

**3.3 PROPOSITION** Suppose  $P$  is a convex polytope of dimension  $n + 1$  and  $M$  is the cell-complex consisting of all faces of  $P$  which are of dimension  $\leq n$ . If  $P$  is simple,  $M$  is a simple  $n$ -manifold and is subject to 3.1. If  $P$  is simplicial,  $M$  is an Eulerian  $n$ -manifold and is subject to 3.2.

**Proof.** First verify that  $M$  is a cell-complex; then clearly  $M$  is simplicial if and only if  $P$  is simplicial. It follows by polarity that  $M$  is a simple  $n$ -manifold when  $P$  is simple and then by a second use of polarity that  $M$  is actually an Eulerian  $n$ -manifold when  $P$  is simplicial. ||

Now let  $f(\mathbb{E}^n) := \{f(M) : M \in \mathbb{E}^n\} \subset \mathbb{R}^{n+1}$ , where

$$f(M) := (f_0(M), \dots, f_n(M)) \in \mathbb{R}^{n+1}.$$

Theorem 3.2 implies that both when  $n = 2u - 1$  and when  $n = 2u - 2$ , the set  $f(\mathbb{E}^n)$  lies in a  $u$ -dimensional linear subspace of  $\mathbb{R}^{n+1}$ . Our final result shows that in fact the linear span of  $f(\mathbb{E}^n)$  is  $u$ -dimensional, even when attention is restricted to those Eulerian  $n$ -manifolds which arise from  $(n + 1)$ -dimensional convex polytopes.

3.4 PROPOSITION For  $0 \leq r \leq n + 1$ , let  $C_r^n$  denote the Eulerian  $n$ -manifold which is the join of the boundary  $B_r$  of an  $r$ -simplex and the boundary  $B_{n+1-r}$  of an  $(n + 1 - r)$ -simplex.

When  $n = 2u - 1$  the matrix

$$\begin{pmatrix} f(C_1^n) \\ f(C_2^n) \\ \vdots \\ f(C_u^n) \end{pmatrix}$$

is of rank  $u$ , and when  $n = 2u - 2$  the matrix

$$\begin{pmatrix} f(C_0^n) \\ f(C_1^n) \\ \vdots \\ f(C_{u-1}^n) \end{pmatrix}$$

is of rank  $u$ .

Proof. Each  $s$ -simplex of  $C_r^n$  is the join of a  $(\lambda - 1)$ -simplex (determined by  $\lambda$  vertices) of  $B_r$  and a  $(\mu - 1)$ -simplex (determined by  $\mu$  vertices) of  $B_{n+1-r}$ , where  $\lambda \in [0, r]$ ,  $\mu \in [0, n + r - 1]$ , and  $\lambda + \mu = s + 1$ ; conversely, each such join is an  $s$ -face of  $C_r^n$ . Hence with  $f_{rs}^n = f_s(C_r^n)$  we have

$$f_{rs}^n = \sum_{\lambda \in [0, r], \mu \in [0, n+1-r], \lambda+\mu=s+1} \binom{r+1}{\lambda} \binom{n+2-r}{\mu}.$$

Considering the expansion of the polynomial  $(1+x)^{r+1}(1+x)^{n+2-r} = (1+x)^{n+3}$ , we see that

$$\sum_{\lambda \geq 0, \mu \geq 0, \lambda+\mu=s+1} \binom{r+1}{\lambda} \binom{n+2-r}{\mu} = \binom{n+3}{s+1}.$$

It follows that  $f_{rs}^n = \binom{n+3}{s+1}$  whenever  $\min(r, n+1-r) \geq s$  (and, in particular, when  $n+1 \geq 2r \geq 2s$ ), while  $f_{ss}^n = \binom{n+3}{s+1} - 1$  when  $n+1 < 2s$ .

Now suppose  $n = 2u - 1$  and consider the  $u \times u$  matrix

$$(f_{rs}^n) \quad (1 \leq r \leq u, 0 \leq s \leq u-1).$$

Each element of its 0-column is equal to  $n+3$ , its 1-column starts with  $\binom{n+3}{2} - 1$  and has  $\binom{n+3}{2}$  thereafter;...; its  $s$ -column has  $f_{ss}^n = \binom{n+3}{s+1} - 1$  but  $\binom{n+3}{s+1}$  thereafter. Subtracting the last row from each of the others, we obtain a matrix  $(g_{rs}^n)$  in which the 0-column ends with  $n+3$  but has all its other entries equal to 0, while the matrix

$$(g_{rs}^n) \quad (1 \leq r \leq u-1, 1 \leq s \leq u-1)$$

is triangular, with all 0's below its main diagonal and all -1's along the main diagonal. Hence the determinant of  $(g_{rs}^n)$  is equal to  $n + 3$  and we have the desired conclusion for the case  $n = 2u - 1$ .

Suppose, finally, that  $n = 2u - 2$  and note that since, for each  $r$ ,

$$\sum_{s=0}^n (-1)^s f_s(C_r^n) = \chi(C_r^n) = 2,$$

the rank of the matrix with which we are concerned is not changed by adding a column of 1's. The augmented matrix has the  $u \times u$  submatrix

$$\begin{pmatrix} 1 & f_{00}^n & f_{01}^n & \cdot & \cdot & \cdot & f_{0(u-2)}^n \\ 1 & f_{10}^n & f_{11}^n & \cdot & \cdot & \cdot & f_{1(u-2)}^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & f_{(u-1)0}^n & f_{(u-1)1}^n & \cdot & \cdot & \cdot & f_{(u-1)(u-2)}^n \end{pmatrix},$$

whose determinant is equal to 1 (as is verified by the method employed above). This completes the proof of 3.4. ||

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